Christ Church
Grammar School

2019
TEST 1

MATHEMATICS SPECIALIST Year 12
Calculator-assumed

Your name SOLUTIONS

Teacher's name ____________________

Time and marks available for this section
Reading Time:	5 minutes
Working time for this section:	45 minutes
Marks available:	44 marks

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: draw instruments, templates and up to three calculators approved for use in the WACE examinations

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. The rules of conduct of the CCGS assessments are detailed in the Reporting and Assessment Policy. Sitting this assessment implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. Answer all questions.

4. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

5. Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

6. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

7. It is recommended that **you do not use pencil**, except in diagrams.

Question 1 **(7 marks)**

Given the complex numbers $z_1 = 2 - i$, $z_2 = i$, $z_3 = 2ai$ and $z_4 = p - 6i$, find:

(a) $z_1\overline{z_4}$ (2 marks)

$(2-i)(p+6i)$

$= 2p + 12i - ip - 6i^2$ ✓ expands brackets correctly

$= 2p + 6 + i(12 - p)$ ✓ correct simplification

$(\text{or } 2p + 6 + 12i - pi)$ accept either

(b) $|z_1 + z_3|$ (2 marks)

$\sqrt{2^2 + (-1+2a)^2}$ ✓ correct substitution

$= \sqrt{4 + (2a-1)^2}$ ✓ simplifies correctly

or (accept either solution).

$\sqrt{4a^2 - 4a + 5}$

(c) $arg\left(\frac{z_3}{2az_2}\right)$ (2 marks)

$= \arg\left(\frac{2ai}{2ai}\right)$

$= \arg(1)$ ✓ substitution and correct simplification

$= 0$ ✓ evaluates correctly.

(d) the value of p, such that $z_1\overline{z_4}$ is real. (1 mark)

If $z_1\overline{z_4}$ is real then imag part $= 0$

So $12 - p = 0$

$p = 12$ ✓ correct evaluation of imag part $= 0$ from part (a).

Question 2 **(8 marks)**

Let $a = 1 + i$ and $b = 1 + i\sqrt{3}$.

(a) Express a and b in exact polar form. (2 marks)

$\theta_a = \tan^{-1}(1/1)$
$= \pi/4$
$|a| = \sqrt{1^2+1^2}$
$= \sqrt{2}$
$a = \sqrt{2}\,\text{cis}(\pi/4)$ ✓

$\theta_b = \tan^{-1}(\sqrt{3}/1)$
$= \pi/3$
$|b| = \sqrt{1^2+(\sqrt{3})^2}$
$= 2$
$b = 2\,\text{cis}\,\pi/3$ ✓

no working required

correct polar form for 'a' and 'b'

(b) Find $\frac{b}{a}$ in exact polar form. (1 mark)

$$\frac{2\,\text{cis}\,\pi/3}{\sqrt{2}\,\text{cis}\,\pi/4}$$
$$= \sqrt{2}\,\text{cis}\,\pi/12$$ ✓

evaluates correct magnitude and evaluates correct angle.

(no working required)

(c) Find $\frac{b}{a}$ in exact Cartesian form. (2 marks)

$$\frac{(1+\sqrt{3}i)}{(1+i)} \times \frac{(1-i)}{1-i}$$
$$= \frac{1 - i + \sqrt{3}i - \sqrt{3}i^2}{1-i^2}$$
$$= \frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i$$ ✓

✓ multiplies by conjugate

correct evaluation of real and imaginary parts.

(d) Use your answers from (b) and (c) to find $\cos\frac{\pi}{12}$ in exact simplified form. (3 marks)

$\cos\pi/12$ is real part of cis/polar form.

$$\therefore \sqrt{2}\cos\pi/12 = \frac{1+\sqrt{3}}{2}$$ ✓
$$\cos\pi/12 = \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$ ✓
$$= \frac{\sqrt{2}+\sqrt{6}}{4}$$ ✓

equate real parts of polar form and Cartesian form

rationalising denominator

correct simplification.

Question 3 **(5 marks)**

Let $w = x + yi$.

If $\left|\frac{w-6i}{w-2}\right| = 2$, show that w is represented by a point on a circle and determine the centre and radius of this circle.

$|w-6i| = 2|w-2|$ ✓ correct statement

$x^2 + (y-6)^2 = 2^2\left((x-2)^2 + y^2\right)$ ✓ correct squaring

$x^2 + y^2 - 12y + 36 = 4x^2 - 16x + 16 + 4y^2$

$0 = 3x^2 - 16x + 3y^2 + 12y + 16 - 36 \quad \div 3$

$0 = x^2 - \frac{16}{3}x + y^2 + 4y - \frac{20}{3}$ ✓ dividing by coefficient/value to get ('x^2', 'y^2')

$0 = \left(x - \frac{16}{6}\right)^2 - \frac{64}{9} + (y+2)^2 - 4 - \frac{20}{3}$ can get mark if simplifying incorrect.

$0 = \left(x - \frac{8}{3}\right)^2 + (y+2)^2 - \frac{160}{9}$

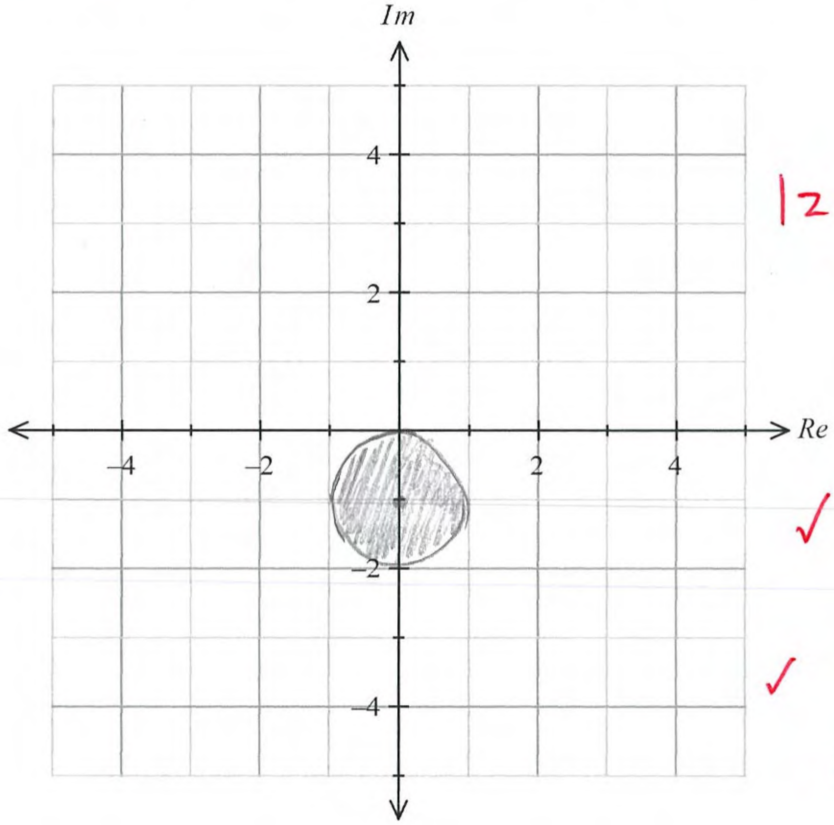
$\frac{160}{9} = \left(x - \frac{8}{3}\right)^2 + (y+2)^2$ ✓ evaluates correct equation (from their result)

$\therefore r = \sqrt{\frac{160}{9}}$ centre $\left(\frac{8}{3}, -2\right)$

$= \frac{4\sqrt{10}}{3}$ ✓ states correct radius and centre. (based on their result).

Question 4 **(6 marks)**

(a) Sketch the region in the Argand Plane defined by $\{z : |z + i| \leq 1\}$. (2 marks)

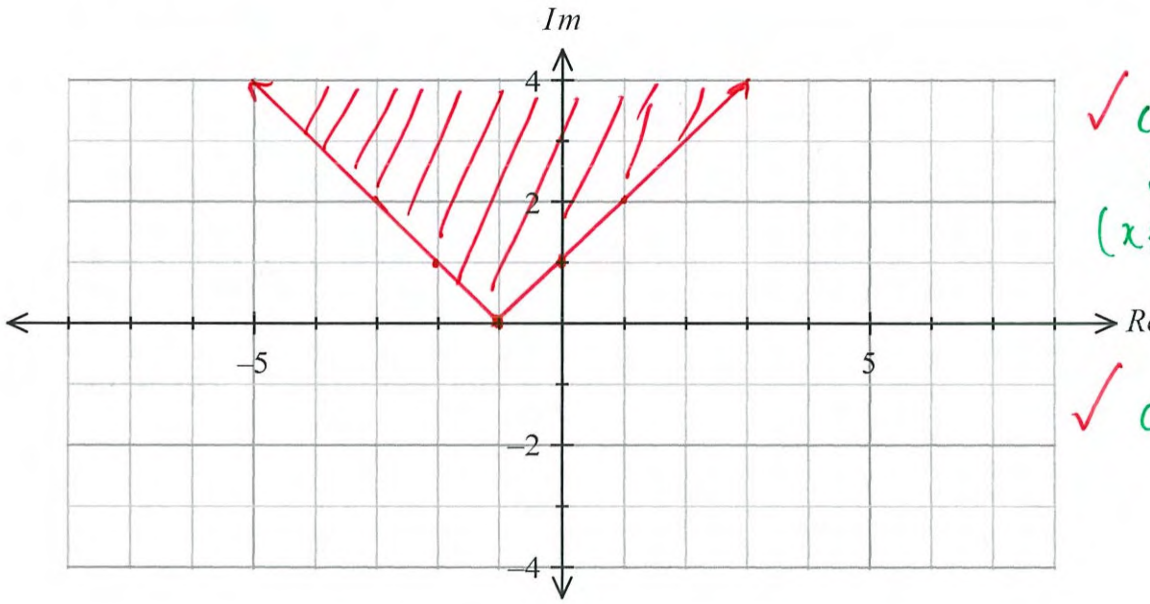


$|z-(-i)| \leq 1$
centre $(0,-1)$
$r = 1$

✓ shows correct centre & radius

✓ correct shape + shading (touches at $(0,-2)$, $(-1,-1)$, $(0,0)$ $(1,-1)$)

(b) Sketch the region in the Argand Plane defined by
$\{z : Im(z) \geq |Re(z) + 1|\}$ (2 marks)

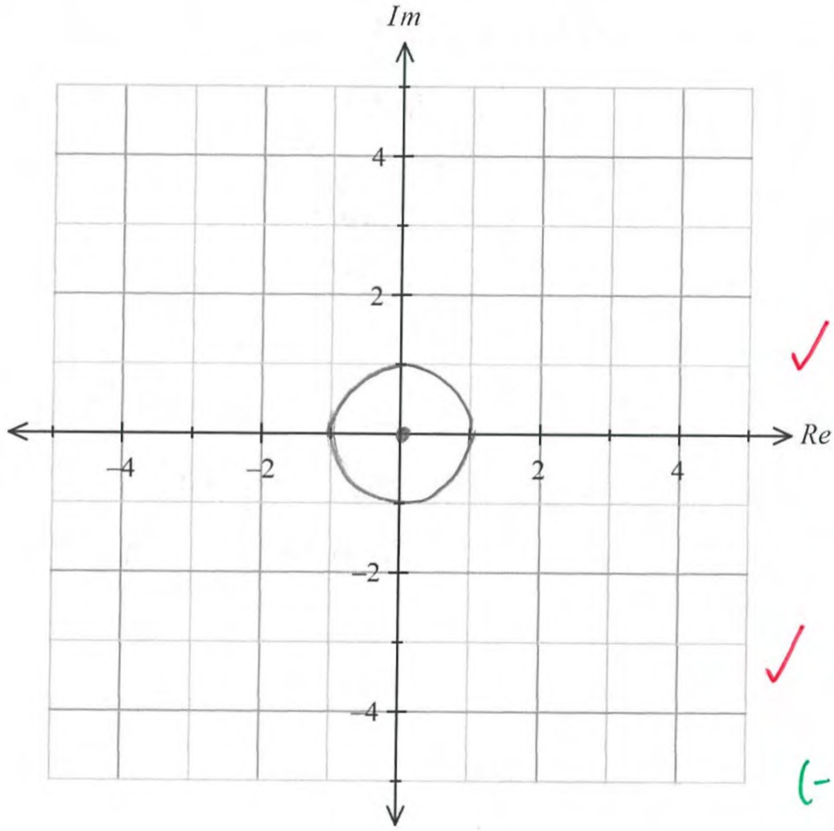


✓ correct graph (x+y intercepts)

✓ correct shading

Question 4 continued

(c) Sketch the region in the Argand Plane defined by $\{z : z = \cos(\theta) + i\sin(\theta) \text{ where } -\pi \leq \theta \leq \pi\}$. (2 marks)



✓ correct centre and radius (implies recognises its a circle)

✓ correct values and shape. (-1,0) (1,0) (0,1) (0,-1)

$x + yi = \cos\theta + i\sin\theta$

$x^2 + y^2 = 1$

$\therefore$ centre $(0,0)$ $r = 1$

Question 5 **(13 marks)**

(a) If $z = cis\,\theta$ show that $z^n + \frac{1}{z^n} = 2\cos n\theta$. (3 marks)

$z = cis\,\theta$

$\therefore z^n = (cis\,\theta)^n$

$= cis(n\theta)$

$z^{-n} = (cis\,\theta)^{-n}$

$= cis(-n\theta)$

$z^n + \frac{1}{z^n} = z^n + z^{-n}$

$= cis(n\theta) + cis(-n\theta)$ ✓ correct substitution

$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$ ✓ correct expansion

$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$ ✓ correct use of rule $i\sin(-n\theta) = -i\sin(n\theta)$

$= 2\cos(n\theta)$

(b) Hence, or otherwise, explain why $z + \frac{1}{z} = 2\cos\theta$. (1 mark)

$n=1$ then $z^1 + \frac{1}{z^1} = 2\cos(1\theta)$

$= 2\cos\theta$ ✓

valid explanation

(c) Expand $\left(z + \frac{1}{z}\right)^3$ and simplify your result. (2 marks)

$= z^3 + 3z^2 \cdot \frac{1}{z} + 3z \cdot \frac{1}{z^2} + \frac{1}{z^3}$ ✓ expands correctly

$= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$ ✓ simplifies correctly

or $(2\cos\theta)^3$

$= 2^3\cos^3\theta$

$= 8\cos^3\theta$

(2 marks for correct answer. No working required)

Question 6 **(5 marks)**

(a) Solve $z^4 + 4 = 0$, writing your solutions in Cartesian form. (2 marks)

$z^4 = -4$

$= 4 cis \pi$

$z_1 = 1 + i$

$z_2 = -1 + i$

$z_3 = -1 - i$

$z_4 = +1 - i$

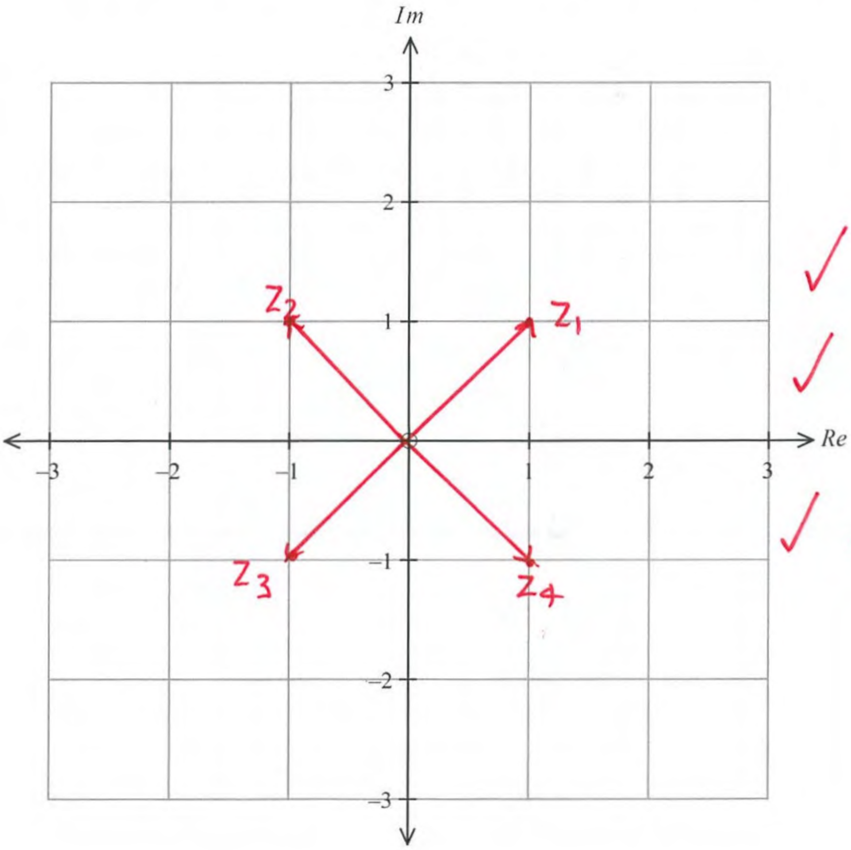
✓✓

2 marks for all 4 correct solutions

-1 if solns in polar form

(ie $z = \sqrt{2} cis \frac{\pi}{4}, \sqrt{2} cis \frac{3\pi}{4}, \sqrt{2} cis -3\pi/4, \sqrt{2} cis -\frac{3\pi}{4}$)

(b) Hence, or otherwise, display the solutions to $z^4 + 4 = 0$, on an Argand diagram. (3 marks)



✓ correct modulus of all

✓ correct pos^n of all roots (in quadrants)

✓ correct argand diagram.

End of questions

Question 5 continued

(d) Hence, or otherwise, show that $\cos^3\theta = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta)$. (3 marks)

or

$\left(z+\frac{1}{z}\right)^3$

$= z^3 + \frac{1}{z^3} + 3\left(z+\frac{1}{z}\right)$

$= 2\cos(3\theta) + 3(2\cos\theta)$

$= 2\cos(3\theta) + 6\cos\theta$

$\Rightarrow$

$8\cos^3\theta = 2\cos3\theta + 6\cos\theta$

$\cos^3\theta = \frac{1}{4}\cos^3\theta + \frac{3}{4}\cos\theta \Rightarrow$

$\text{cis}^3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos3\theta + i\sin3\theta$

and

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

So $\cos3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ ✓ equating real parts

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$ ✓ subst. in $(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$ ✓

$\cos3\theta + 3\cos\theta = 4\cos^3\theta$ ✓ correct simplifying

$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos3\theta$

(e) Hence, show the exact value of $\cos^3\left(\frac{13\pi}{12}\right) = \frac{-5\sqrt{2}-3\sqrt{6}}{16}$. (4 marks)

Hint: $\frac{13\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3}$

$\cos^3\theta = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos\theta$

$\cos^3\left(\frac{13\pi}{12}\right) = \frac{1}{4}\cos\left(3\left(\frac{3\pi}{4}+\frac{\pi}{3}\right)\right) + \frac{3}{4}\cos\left(\frac{3\pi}{4}+\pi/3\right)$ ✓ substitutes in hint.

$= \frac{1}{4}\cos\left(\frac{9\pi}{4}+\pi\right) + \frac{3}{4}\cos\left(\frac{3\pi}{4}+\pi/3\right)$ expands using double angle ✓

$= \frac{1}{4}\left(\cos\frac{\pi}{4}\cos\pi - \sin\frac{\pi}{4}\sin\pi\right) + \frac{3}{4}\left(\cos\frac{3\pi}{4}\cos\frac{\pi}{3} - \sin\frac{3\pi}{4}\sin\frac{\pi}{3}\right)$

$= \frac{1}{4}\left(\frac{1}{\sqrt{2}}\cdot -1 - \frac{1}{\sqrt{2}}\cdot 0\right) + \frac{3}{4}\left(-\frac{1}{\sqrt{2}}\cdot\frac{1}{2} - \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}\right)$ ✓

$= \left(-\frac{1}{4\sqrt{2}} - 0 + -\frac{3}{8\sqrt{2}} - \frac{3\sqrt{3}}{8\sqrt{2}}\right) \times\frac{\sqrt{2}}{\sqrt{2}}$ evaluates ~~value~~ angles (must show)

$= -\frac{\sqrt{2}}{8} - \frac{3\sqrt{2}}{16} - \frac{3\sqrt{6}}{16}$

$= \frac{-2\sqrt{2} - 3\sqrt{2} - 3\sqrt{6}}{16}$ ✓ simplifies

$= \frac{-5\sqrt{2} - 3\sqrt{6}}{16}.$

Additional working space

Question number: __________

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